EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. A cubic f(x) vanishes at x = –2 & has relative minimum/maximum at x = –1 and x = 1/3. If $\int_{-1}^{1} f(x)dx = \frac{14}{3}$, find the cubic f(x).

2. Find the greatest & least value for the function ;

(a) $y = x + \sin 2x,\ 0 \le x \le 2\pi$

(b) $y = 2\cos 2x - \cos 4x,\ 0 \le x \le \pi$

3. Suppose f(x) is real valued polynomial function of degree 6 satisfying the following conditions ;

(a) f has minimum value at x = 0 and 2

(b) f has maximum value at x = 1

(c) for all x, $\lim_{x\to 0} \frac{1}{x} \ell n \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2$. Determine f(x).

4. Find the maximum perimeter of a triangle on a given base 'a' and having the given vertical angle α.

5. The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

6. The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's'. If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum.

7. For a given curved surface of a right circular cone when the volume is maximum, prove that the semi vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

8. Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

9. A statue 4 meters high sits on a column 5.6 meters high . How far from the column must a man, whose eye level is 1.6 meters from the ground, stand in order to see the statue at the greatest angle ?

10. By the post office regulations, the combined length & width of a parcel must not exceed 3 meter . Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post .

11. A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end . If the area of the rectangular portion is to be maximum, find the length of its sides.

12. A closed rectangular box with a square base is to be made to contain 1000 cubic feet . The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise . The labour charges for making the box are Rs. 3/- . Find the dimensions of the box when the cost is minimum .

13. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

14. A trapezium ABCD is inscribed into a semicircle of radius l so that the base AD of the trapezium is a diameter and the vertices B & C lie on the circumference. Find the base angle θ of the trapezium ABCD which has the greatest perimeter.

15. If $y = \frac{ax+b}{(x-1)(x-4)}$ has a turning value at (2, –1) find a & b and show that the turning value is a maximum.

16. Prove that among all triangles with a given perimeter, the equilateral triangle has the maximum area.

17. A sheet of poster has its area 18 m² . The margin at the top & bottom are 75 cms and at the sides 50 cms . What are the dimensions of the poster if the area of the printed space is maximum ?

18. A perpendicular is drawn from the centre to a tangent to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.

19. Consider the function, $F(x)=\int_{-1}^{x}(t^2 - t)\,dt,\ x \in R$.

(a) Find the x and y intercept of F if they exist.
(b) Derivatives $F'(x)$ and $F''(x)$.
(c) The intervals on which F is an increasing and the intervals on which F is decreasing.
(d) Relative maximum and minimum points.
(e) Any inflection point.

20. A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression ; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

21. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

22. The flower bed is to be in the shape of a circular sector of radius r & central angle θ. If the area is fixed & perimeter is minimum, find r and θ.

23. The mass of a cell culture at time t is given by ,

$M(t) = \dfrac{3}{1+4e^{-1}}$

(a) Find $\lim\limits_{t\to-\infty} M(t)$ and $\lim\limits_{t\to\infty} M(t)$

(b) Show that $\dfrac{dM}{dt}=\dfrac{1}{3}M(3 - M)$

(c) Find the maximum rate of growth of M and also the value of t at which occurs.

24. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length ℓ of the median drawn to its lateral side.

25. Find the fixed point A on the circumference of a circle of radius 'a', let the perpendicular AY fall on the tangent at a point P on the circle, prove that the greatest area which the ΔAPY can have is $3\sqrt{3}\dfrac{a^2}{8}$ sq. units.

26. Given two points A (– 2 , 0) & B (0 , 4) and a line y = x . Find the co-ordinates of a point M on this line so that the perimeter of the Δ AMB is least.

27. A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends . Show that in order that total surface area may be minimum , the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi + 2)$.

28. Depending on the values of $p \in R$, find the value of 'a' for which the equation $x^3 + 2px^2 + p = a$ has three distinct real roots.

29. Show that for each $a > 0$ the function $e^{-ax}.\, x^{a^2}$ has a maximum value say F(a) , and that F(x) has a minimum value, $e^{-e/2}$.

30. For a > 0, find the minimum value of the integral $\int_{0}^{1/a}(a^3 + 4x - a^5 x^2)e^{ax}\,dx$.

31. Consider the function $f(x) = \begin{cases}\sqrt{x}\,\ell n x & \text{when } x > 0\\ 0 & \text{for } x = 0\end{cases}$

(a) Find whether f is continuous at x = 0 or not.
(b) Find the minima and maxima if they exist.

(c) Does $f'(0)$? Find $\lim\limits_{x\to0} f'(x)$

(d) Find the inflection points of the graph of y = f(x).

32. Consider the function y = f(x) = *l*n (1 + sin x) with $-2\pi \le x \le 2\pi$. Find
(a) the zeroes of f(x)
(b) inflection points if any on the graph
(c) local maxima and minima of f(x)
(d) asymptotes of the graph
(e) sketch the graph of f(x) and compute the value of the definite integral $\int_{-\pi/2}^{\pi/2} f(x)\,dx$.

33. A right circular cone is to be circumscribed about a sphere of a given radius. Find the ratio of the altitude of the cone to the radius of the sphere, if the cone is of least possible volume.

34. Find the set of value of m for the cubic

$x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

35. A cylinder is obtained by revolving a rectangle about the x–axis, the base of the rectangle lying on the x–axis and the entire rectangle lying in the region between the curve. $y = \frac{x}{x^2+1}$ & the x–axis. Find the maximum possible volume of the cylinder.

36. The value of 'a' for which
$f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

37. Among all regular triangular prism with volume V, find the prism with the least sum of lengths of all edges. How long is the side of the base of that prism ?

38. What is the radius of the smallest circular disk large enough to cover every acute isosceles triangle of a given perimeter L ?

39. Find the magnitude of the vertex angle 'α' of an isosceles triangle of the given area 'A' such that the radius 'r' of the circle inscribed into the triangle is the maximum.

40. The function f(x) defined for all real numbers x has the following properties
$f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find
(a) the intervals on which f is increasing and decreasing and any local maximum or minimum values.
(b) the intervals on which the graph f is concave down and concave up.
(c) the function f(x) and plot its graph.

41. Use calculus to prove the inequality, $\sin x \geq 2x/\pi$ in $0 \leq x \leq \pi/2$.
You may use the inequality to prove that, $\cos x \leq 1 - x^2/\pi$ in $0 \leq x \leq \pi/2$.